MATH 579: Combinatorics Final Exam

Please read the following instructions. For the following exam you may use a calculator, and one page of notes. You may not use any other papers, books, or computers. Please turn in **exactly nine** problems. You must do problems 1-6, and three more chosen from 7-10. Please write your answers on separate paper, make clear what work goes with which problem, adequately justify all answers, simplify all numerical answers as best you can, and put your name or initials on every page. You have 120 minutes. Each problem will be graded on a 5-10 scale (as your quizzes), for a total score between 45 and 90. This will then be multiplied by $\frac{10}{9}$ for your exam score.

Turn in problems 1-6:

1. We fill each entry of a 3×3 grid with a letter, chosen from $\{a,b,c,d,e\}$, subject to the condition that each row and each column must have three distinct letters. Prove that there are at least 8641 such grids.
2. Find the number of solutions in integers to $x_1+x_2+x_3+x_4+x_5=20$, satisfying $x_1,x_2,x_3,x_4\geq 2$ and $4\leq x_5\leq 6$.
3. Let $a,b\in\mathbb{N}_0$. Prove that $\binom{x}{a}\binom{x-a}{b}=\binom{x}{b}\binom{x-b}{a}$.
4. Use difference calculus to compute $\sum_{i=1}^{10} 3^i-2^i-i^2$.
5. Solve the recurrence given by $a_0=5, a_1=14, a_n=2a_{n-1}+3a_{n-2}+4(-1)^n$ $(n\geq 2)$.
6. The complete tripartite graph $K_{r,g,b}$ contains r red vertices, g green vertices, and b blue vertices. It contains every possible edge between two vertices of different colors, and no edges between vertices of the same color. Use Burnside's lemma to count the number of ways to color the vertices of $K_{2,2,3}$, distinct up to graph automorphism, drawn from 2 possible colors.

Turn in exactly three more problems of your choice:

7. Let $k\in\mathbb{N}_0$. Prove that $\binom{\frac{1}{2}}{k}=(-1)^{k+1}\binom{2k}{k}\frac{2^{-2k}}{2k-1}$.
8. Prove that, for all $n\in\mathbb{N}$, $p(n)^2<p(n^2+2n)$.
9. Let G be a graph on n vertices. Suppose that, for each pair of distinct vertices x,y, either (a) there is an edge from x to y; or (b) $\deg(x)+\deg(y)\geq n-1$. Prove that G is connected. Note: $\deg(v)$ denotes the degree of vertex v.
10. For $n,k\in\mathbb{N}_0$, the Stirling number of the first kind $s(n,k)$ counts the number of permutations of $[n]$ (i.e. elements of S_n) with exactly k cycles. Prove that, for all $n,k\in\mathbb{N}$,
$$s(n,k)=(n-1)s(n-1,k)+s(n-1,k-1).$$

Please keep this sheet for your records.